RESONANT AND NONRESONANT CASE IN THE PROBLEM OF EXCITATION OF MECHANICAL OSCILLATIONS

PMM Vol. 32, No. 1, 1968, pp. 36-45

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(*Received August 31, 1967*)

We consider the problem of excitation of mechanical oscillations in linear oscillatory systems. We show that the solution obtained by the method of small parameter for the case when resonance is absent in the oscillating system, can be used to find the resonant oscillations. We also propose a generalization of the Malkin-Shimanov theorem on periodic solutions of quasilinear systems of ordinary differential equations ([1], Chapt. 2, Section 9) extending this theorem to the problems of excitation of oscillations in the systems with distributed parameters.

1. When an oscillating system exhibits displacements which are small compared with the characteristic dimension of the exciter (such are the problems of dynamics of systems with mechanical vibrators [2 to 4]), then the equations of the problems of excitation of mechanical oscillations contain a small parameter and can therefore be written as

$$\varphi^{\cdot} = \Phi(\varphi, t) + \mu\Theta(\varphi, \xi, \xi^{\cdot}, \xi^{\cdot\cdot}, t, \mu),$$

$$Mu^{\cdot\cdot} + \gamma Bu^{\cdot} + Cu = f\Big[\sum_{r=1}^{m} Q_r(\varphi, \varphi^{\cdot})\, q_r + \mu \ldots\Big] \qquad (1.1)$$

Here $\phi = (\phi_1, \ldots, \phi_k)$ is a vector whose components are used to describe the work (motion) of the exciters, u is a N-dimensional vector or an element of a Hilbert space H characterizing the configuration of the oscillatory system; M, B and C are $N \times N$ matrices whose components either are time-independent or are linear operators in H; $q_1, \ldots, q_m$ are either given constant vectors or elements of H describing the distribution of forces generated by the exciters over the system; $\xi = (\xi_1, \ldots, \xi_m)$ $\xi_r = (u, q_r)$, $r = 1, \ldots, m$; a bracket denotes a scalar product and f, $\gamma > 0$ and $\mu > 0$ are scalar parameters, the last of which is assumed to be sufficiently small.

Equations of the problem of excitation of mechanical oscillations can be reduced to (1.1) also in some of the cases when the displacements indicated above are of the order of the typical dimension of the exciters. Problems on the oscillations generated by electromagnets [5] are an example of such a case.

We assume that the terms of equations of motion which describe the feedback effect of the generated oscillations on the exciters and which enter $\Theta(\phi, \xi, \ldots)$, are not direct functions of the coordinate u. They depend on the magnitudes $\xi_1, \ldots, \xi_m$ which shall be called the "feedback parameters". This can be explained by the fact that the physical interpretation of $\xi_1, \ldots, \xi_m$ and the form of Θ and Q_r are usually independent of the form assumed by the oscillating system and of the method of introduction of the coordinate u. The latter only define the vectors q_r. In the case of electromagnets, the distances between their armatures and cores [5] become the feedback parameters, while in the case of mechanical vibrators, the displacements of their axes [4].

In this paper we shall consider the case for non-self-contained systems only, although our arguments shall also be valid in the case of self-contained systems. The right-hand sides of Eqs. (1.1) are assumed to be 2π-periodic in time, which enters the equations explicitly. Usual assumptions [1] about their smoothness are made, and we consider 2π-periodic solutions.

When oscillations appear in a linear oscillatory system under the action of an exciter, we must consider two cases, resonant and nonresonant, separately. They will correspond to different properties of the solution of the problem on the forced oscillations of the oscillatory system when the drivind forces are given. Let us consider the case, when an applied load is 2π-periodic and distributed over the system according to the law defined in the terms of one of the vectors q_r

$$Mu_r^{\cdot\cdot} + \gamma Bu_r^{\cdot} + Cu_r = F(t)\, q_r \tag{1.2}$$

We shall calculate the feedback parameters $\xi_{r1},\ldots, \xi_{rm}$ for the 2π-periodic solution of this equation. When the system is nonresonant then we should have, for any sufficiently smooth $F(t)$, $\varkappa_{rs} = (\max|\xi_{rs}|)$: $(\max|F(t)|) = O(1)$ for $r, s = 1,\ldots, m$. In the resonant case $\varkappa_{rs} = O(1/\mu)$ for at least one pair of values r, s. Moreover, in accordance with the physical demands of the problem the parameter f should, in the nonresonant case, be not small i.e. $f = O(1)$, while in the resonant case $f = O(\mu)$. Let us now assume that the given 2π-periodic solutions

$$\varphi^{(0)} = \varphi^{(0)}(t,\ \alpha),\ \ \varphi^{(0)} = (\varphi_1^{(0)},\ \ldots,\ \varphi_k^{(0)}),\ \alpha = (\alpha_1,\ldots,\alpha_j)$$

of the system

$$\varphi^{(0)\prime\prime} = \Phi(\varphi^{(0)},\ t) \tag{1.3}$$

form a family with j constants $\alpha_1,\ldots, \alpha_j$ (the case of an isolated generating solution is of little interest in the study of forced oscillations; we should also note that, when the oscillatory forces are small, then only the generating solution needs to be determined [4]). Let us suppose as well that 2π-periodic solutions $z_{1i},\ldots, z_{ki}$ $(i = 1,\ldots, j)$ of a system conjugate to the system

$$z_\beta^{\cdot} + p_{1\beta}z_1 + \ldots + p_{k\beta}z_k = 0 \quad (\beta = 1,\ldots,n),\ p_{r\beta} = \left(\frac{\partial \Phi_r}{\partial \varphi_\beta}\right)_{\varphi=\varphi^{(0)}} \quad (r,\ \beta = 1,\ \ldots, k) \tag{1.4}$$

written in the terms of variations of (1.3) are known, and that there are exactly j solutions. We shall utilize the fact that the functions Φ, Θ and Q_r depend only on properties of the exciters, and not on the form of the oscillating system. Then, expressions for the parameters of the generating solution $\alpha_1,\ldots, \alpha_j$ and driving forces $Q_r^{(0)}$ computed in accordance with the generating solution as well as the stability conditions, can be presented for the nonresonant case solely in the terms of components of the amplitude and phase frequency matrix characteristics of the oscillating system. This can be done as follows.

Inserting (1.3) into $Q_r(\phi, \phi^{\cdot})$ and expanding the latter into trigonometric polynomials (as in [4 and 5]) or (in the general case) into the Fourier series, we obtain

$$Q_r^{(0)}(t,\alpha) = \sum Q_{r\nu}^{(0)}(\alpha)\cos(\nu t - \vartheta_{r\nu}(\alpha)) \qquad (r = 1,\ldots,m) \tag{1.5}$$

Let us now introduce the matrices $K_\nu = \|k_\nu^{(rs)}\|$ and $\Psi_\nu = \|\psi_\nu^{(rs)}\|$ defining the amplitudes and phase shifts of the feedback parameters when the oscillations generated by known unit harmonic forces of frequency ν, are steady. Magnitudes $k_\nu^{(rs)}$ and $\psi_\nu^{(rs)}$ are given by

$$\xi_\nu^{(rs)} = k_\nu^{(rs)}\cos(\nu t - \psi_\nu^{(rs)}), \qquad \xi_\nu^{(rs)} = (u_\nu^{(r)},\ q_s) \tag{1.6}$$

where $u_\nu^{(r)}$ are $2\pi/\nu$-periodic solutions of

$$Mu_\nu^{(r)\cdot\cdot} + \gamma Bu_\nu^{(r)\cdot} + Cu_\nu^{(r)} = \cos\nu t q_r \tag{1.7}$$

Let us now write the expressions for the feedback parameters. It is

$$\xi_s^{(0)} = \sum_{r=1}^{m} \sum_{\nu} Q_{r\nu}^{(0)}(\alpha)\, f k_\nu^{(rs)} \cos(\nu t - \vartheta_{r\nu}(\alpha) - \psi_\nu^{(rs)}) \quad (s = 1, \ldots, m) \tag{1.8}$$

They are computed from the generating solution and contain the components of the matrices K_ν and Ψ_ν as parameters.

Further, inserting (1.3) and (1.8) into small parts of the first k Eqs. of (1.1) (the resulting vector is denoted here by Θ_0) and setting up the equations defining the parameters of the generating solution in the usual [1] manner, we obtain j equations

$$P_i(\alpha_1, \ldots, \alpha_j, \ldots, fK_\nu, \Psi_\nu, \ldots) \equiv \qquad (i = 1, \ldots, j) \tag{1.9}$$

$$\equiv \frac{1}{2\pi} \int_0^{2\pi} \sum_{\beta=1}^{k} \Theta_{0\beta}(t, \alpha_1, \ldots, \alpha_j, \ldots, fK_\nu, \Psi_\nu, \ldots)\, z_{\beta i}(t, \alpha_1, \ldots, \alpha_j)\, dt = 0$$

where ν assumes either all or some of the values which it takes in (1.5).

Next we obtain the parameters $\alpha_1 = \alpha_1^*, \ldots, \alpha_j = \alpha_j^*$ which form the solution of (1.9) in terms of the components of K_ν and Ψ_ν, substitute them into (1.5). This yields the relations

$$\alpha_i^* = \alpha_i^*(\ldots, fK_\nu, \Psi_\nu, \ldots) \qquad (i = 1, \ldots, j) \tag{1.10}$$

$$\xi_r = \xi_r(t, \ldots, fK_\nu, \Psi_\nu, \ldots), \qquad Q_r^{(0)} = Q_r^{(0)}(t, \ldots, fK_\nu, \Psi_\nu, \ldots) \quad (r = 1, \ldots, m)$$

defining the oscillations generated by the given exciters in any linear oscillating system, provided that the matrices K_ν and Ψ_ν are found for this system for all ν appearing in (1.9). This is equivalent to solving a problem on forced oscillations (the corresponding problem on synchronization of mechanical vibrators was dealt with in [4], while the oscillations generated by electromagnets were investigated in [5]). It now remains to show that the relations obtained can also be used in the resonant case.

2. Matrices K_ν and Ψ_ν can be obtained for both, oscillating systems with a finite number of degrees of freedom and for the systems with distributed parameters, and for this reason the above method of reducing the problem on the excitation of oscillations to the problem on forced oscillations embraces, formally, both types of systems. In the cases however, when the oscillating system falls into the pattern characteristic of the system with distributed parameters, a question arises whether periodic solutions of (1.1) exist and, whether the sequence of approximations usually present in the method of small parameter, converges to them. In this connection we may find useful the following simple generalization of a theorem first proved by Malkin and later, more vigorously, by Shimanov ([1], Chapt. 2, Section 9).

Let us assume that the state of the given physical system is defined by an element u of the linear space U and let m linear functionals $\xi_1(u), \ldots, \xi_m(u)$ be also defined on U. Let also the periodic solution of equations of motion of the system acted upon by given periodic forces (these equations may be partial differential equations, equations with time delay e.a.) generate a correspondence between m given 2π-periodic functions $F_1(t), \ldots, F_m(t)$ (which can be interpreted as loads) and a 2π-periodic function $u(t)$, the correspondence being assumed linear. We shall denote this by $u(t) \leftarrow (F_1(t), \ldots, F_m(t))$. Finally let us assume that the 2π-periodic function $u(t)$ exists and is continuous for any 2π-periodic functions $F_1, \ldots, F_m$ possessing a continuous first order derivative (*).

Following [6] we shall say that the system possessing the above properties has a "weak generalized filter property on the class of functions possessing continuous first order derivatives", if for any $F(t)$ belonging to this class the following inequalities hold

*) Periodic solutions of the problem on forced vibrations of oscillating systems which are of some practical interest, have been obtained for a much wider class of loads. The restriction imposed here is due to the same reasons which caused the author of [1] to limit himself to small order terms in the system studied in Section 9, Chapt. 2 of [1].

$$\max_t | \xi_s \; (u_r (t)) | < h_{rs} \; \max_t | F(t) | \qquad (r, \; s = 1,\ldots,m) \tag{2.1}$$
$$u_r (t) \leftarrow (0,\ldots,F_r \; (t) = F(t),\ldots,0), \qquad h_{rs} > 0$$

where h_{rs} are constants. Let us consider the system

$$\varphi_s' = a_{s1}\varphi_1 + \ldots + a_{sk}\varphi_k + \eta_s (t) + \mu\Theta_s (\varphi_1, \ldots, \varphi_k, \; \xi_1, \ldots, \xi_m, \; t, \; \mu) \qquad (s = 1, \ldots, k$$
$$u (t) \leftarrow (Q_1 (\varphi_1, \ldots, \varphi_k), \ldots, Q_m (\varphi_1, \ldots, \varphi_k)), \quad \xi_1 = \xi_1 (u), \ldots, \xi_m = \xi_m (u) \tag{2.2}$$

where $u(t)$ corresponds to the system possessing the filter property defined above. We shall assume that the functions $Q_1,\ldots, Q_m$ are defined on some region G of the space of variables $\phi_1,\ldots, \phi_k$ and, that they have continuous second order derivatives in all their arguments in this region. Assumptions concerning the smoothness of remaining functions shall be those used in [1], Chapt. 2, Section 9 (with ξ_r assumed equal to ϕ_s) and the closed domain of definition of Θ_s belonging to the space of variables $\phi_1,\ldots, \phi_k, \xi_1,\ldots, \xi_m$ shall be denoted by G_*; functions η_s and Θ_s shall be assumed 2π-periodic in t which appears in them explicitly, and we shall assume that the system

$$\varphi_s^{(0)'} = a_{s1}\varphi_1^{(0)} + \ldots + a_{sk}\varphi_k^{(0)} + \eta_s (t) \qquad (s = 1, \ldots, k) \tag{2.3}$$

admits a family of 2π-periodic solutions with j constants

$$\varphi_s^{(0)} = \varphi_s^{(0)} (t, \; \alpha_1, \ldots, \alpha_j) \tag{2.4}$$

(relevant demands on the coefficients of (2.3) and $\eta_s(t)$ and on the form of solutions are given in [1], Chapt. 2, Section 4). Let us now construct a system of Eqs. (2.5)

$$P_i (\alpha_1, \ldots, \alpha_j) \equiv \frac{1}{2\pi} \int_0^{2\pi} \sum_{\beta=1}^{k} \Theta_\beta (\varphi_1^{(0)}, \ldots, \varphi_k^{(0)}, \; \xi_1^{(0)}, \ldots, \xi_m^{(0)}, \; t, \; 0) \, z_{\beta i} (t) \, dt = 0$$
$$(i = 1, \ldots, j)$$
$$\xi_r^{(0)} (t, \alpha_1, \ldots, \alpha_j) = \xi_r (u^{(0)} (t)), \; u^{(0)}(t) \leftarrow (Q_1 (\varphi_1^{(0)}, \ldots, \varphi_k^{(0)}), \ldots, Q_m (\varphi_1^{(0)}, \ldots, \varphi_k^{(0)}))$$

where the functions $z_{\beta i}$ ($\beta = 1,\ldots, k$; $i = 1,\ldots, j$) form of set of j periodic solutions of a system conjugate to the homogeneous system obtained from (2.3) by putting $\eta_s(t) \equiv 0$.

Then the following generalization of the theorem appearing in Section 9, Chapt. 2 of [1] can be given.

Let the system (2.5) admit the following simple solution:

$$\alpha_1 = \alpha_1^*, \ldots, \alpha_j = \alpha_j^*$$

such that

$$\varphi_{1*}^{(0)}, \ldots, \varphi_{k*}^{(0)} \in G, \qquad \varphi_{1*}^{(0)}, \ldots, \varphi_{k*}^{(0)}, \; \xi_{1*}^{(0)}, \ldots, \xi_{m*}^{(0)} \in G_* \;\; \text{when } t \geqslant 0$$

$$\varphi_{1*}^{(0)} = \varphi_1^{(0)} (t, \; \alpha_1^*, \ldots, \alpha_j^*) \qquad \text{etc.}$$

Then for $0 < \mu \leqslant \mu_0$ where μ_0 is a constant, the system (2.2) admits a 2π-periodic solution such that the corresponding functions $\phi_1(t, \mu),\ldots, \phi_k(t, \mu), \xi_1(t, \mu),\ldots, \xi_m(t, \mu)$ remain in G and G_* when $t \geqslant 0$, become $\phi_{1*}^{(0)}(t),\ldots, \xi_{m*}^{(0)}(t)$ when $\mu = 0$, and such, that the sequences of 2π-periodic functions $\phi_1^{(\rho)}(t, \mu),\ldots, \xi_m^{(\rho)}(t, \mu)$ ($\rho = 1, 2$) defined by the following Eqs.

$$\varphi_s^{(\rho)'} = a_{s1}\varphi_s^{(\rho)} + \ldots + a_{sk}\varphi_k^{(\rho)} + \eta_s (t) + \tag{2.6}$$
$$+ \mu\Theta_s (\varphi_1^{(\rho-1)}, \ldots, \varphi_k^{(\rho-1)}, \; \xi_1^{(\rho-1)}, \ldots, \xi_m^{(\rho-1)}, \; t, \; \mu) \qquad (s = 1, \ldots, k)$$
$$u^{(\rho)} \leftarrow (Q_1 (\varphi_1^{(\rho)}, \ldots, \varphi_k^{(\rho)}), \ldots, Q_m (\varphi_1^{(\rho)}, \ldots, \varphi_k^{(\rho)}))$$

converge uniformly to the solution $\phi_1(t, \mu),\ldots, \xi_m(t, \mu)$.

The proof of the above mentioned theorem will suffice here (it is fairly complex, since the values of the constants appearing in it must be estimated at each stage of the proof); we must however construct the inequalities connecting $|\xi_r^{(\rho)} - \xi_r^{(\rho-1)}|$ with $|\phi_s^{(\rho)} -$

$-\phi_1^{(\rho-1)}|,\ldots,|\phi_k^{(\rho)}-\phi_k^{(\rho-1)}|$ in order to obtain all the necessary estimates. This is possible, since by (2.1) the values of $|\xi_r^{(\rho)}-\xi_r^{(\rho-1)}|$ can be estimated from

$$|Q_s(\varphi_1^{(\rho)},\ldots,\varphi_k^{(\rho)})-Q_s(\varphi_1^{(\rho-1)},\ldots,\varphi_k^{(\rho-1)})| \qquad (s=1,\ldots,m)$$

while the remaining magnitudes are estimated by

$$|\varphi_r^{(\rho)}-\varphi_r^{(\rho-1)}| \qquad (r=1,\ldots,k)$$

utilising the Lipshits conditions for Q_s.

If U has a bound, then we can, in a number of cases, show also that the sequence $u^{(\rho)}$ converges to the solution.

Such theorems make it possible to extend the results obtained when the small parameter method is applied to systems possessing a finite number of degrees of freedom, to the case of systems possessing the filter property elucidated above. Rozenvasser obtained in [6] a number of such results for various approximate methods of determination of periodic solutions, while studying the corresponding integral equations. Our assertion given above shows, that in this sense the method of small parameter in the case of a family of generating solutions, is no exception.

3. The process described in Section 1 by which the solution of the problem on excitation of oscillations is reduced to constructing the relations (1.10) and solving the problem on forced oscillations, cannot be applied to the resonant case. A special resonant procedure, which follows, is required in this case to obtain the periodic solutions. We shall confine ourselves to oscillating systems with a finite number of degrees of freedom. From the previous assumptions it follows that, for the oscillating systems considered in the resonant case, the following relations are valid

$$M=M_0+\mu dM_1,\quad C=C_0+\mu cC_1,\quad \gamma=\mu g,\quad f=\mu f_1 \tag{3.1}$$

where the matrices M_0 and C_0 are such, that the polynomials

$$\Delta_N(\lambda)=\det\|C_0-\lambda M_0\| \tag{3.2}$$

has a number of roots, all of which have values equal to the squares of natural numbers. The system (1.1) will thus become

$$\varphi^{\cdot}=\Phi(\varphi,t)+\mu\,\Theta(\varphi,\xi^{\cdot},\xi^{\cdot\cdot},\xi^{\cdot\cdot},t,\mu) \tag{3.3}$$

$$M_0u^{\cdot\cdot}+C_0u=-\mu\Big[dM_1u^{\cdot\cdot}+gBu^{\cdot}+cC_1u-f_1\sum_{r=1}^{m}Q_r(\varphi,\varphi^{\cdot})\,q_r\Big]+\mu^2\ldots$$

We further assume that at least one of the numbers d, c and g is different from zero; the case $d=c=g=0$ corresponds to the oscillating system without friction tuned exactly to the resonant frequency, and is not of interest.

Let the polynomial $\Delta_N(\lambda)$ have h roots $\nu_1^2,\ldots,\nu_h^2$, their values equal to the squares of natural numbers (each of them counted the number of times equal to their multiplicity) and let the remaining $N-h$ roots differ from the squares of natural numbers by magnitudes of the order of unit μ.

We shall assume that the matrices M, M_0, C, C_0 and B are symmetric, C, C_0 and B are nonnegative, while M and M_0 are positive definite; these assumptions are compatible with the requirements imposed by the physical demands of the problem.

Then, 2π-periodic solutions of the generating system obtained from (3.3) by putting $\mu=0$, will form a family with $j+2h$ constants $a_1,\ldots,a_j$, $A_{11},\ldots,A_{\varkappa\rho_\varkappa}$, $D_{11},\ldots,D_{\varkappa\rho_\varkappa}$. These solutions will be of the form

$$\varphi^{(0)}=\varphi^{(0)}(t,\alpha),\qquad u^{(0)}=\sum_{n=1}^{\varkappa}\sum_{\rho=1}^{\rho_n}(A_{n\rho}\cos\nu_nt+D_{n\rho}\sin\nu_nt)\,u_{n\rho} \tag{3.4}$$

where ρ_n denotes the multiplicity of the root $\lambda=\nu_n^2$ of the polynomial $\Delta_N(\lambda)$; $\varkappa$ is the number of different roots of the given type and, obviously, $\rho_1 + \ldots + \rho_\varkappa = h$. We should note that under these assumptions the multiple roots of $\Delta_N(\lambda)$ have the corresponding linear elementary divisors [7], while the eigenvectors $u_{11}, \ldots, u_{\varkappa\rho_\varkappa}, u_{h+1}, \ldots, u_N$, where the last $N-h$ vectors correspond to the roots differing from the squares of natural numbers, form the basis on the configurational space of the oscillating system. This basis is assumed to be orthonormalised, in the sense that

$$(M_0 u_\rho,\ u_\varkappa) = \delta_{\rho\varkappa} \tag{3.5}$$

where u_ρ and $u_\varkappa$ are any two eigenvectors and $\delta_{\rho\varkappa}$ is the Kronecker delta.

First j equations defining the parameters of the generating resonant solution are constructed analogously to (1.9). They have the form

$$P_i^*(\alpha_1, \ldots, \alpha_j, A_{11}, \ldots, A_{\varkappa\rho_\varkappa}, D_{11}, \ldots, D_{\varkappa\rho_\varkappa}) \equiv \qquad (i = 1, \ldots, j) \tag{3.6}$$
$$\equiv \frac{1}{2\pi}\int_0^{2\pi} \sum_{\beta=1}^{k} \Theta_{0\beta}^*(t, \alpha_1, \ldots, \alpha_j, A_{11}, \ldots, A_{\varkappa\rho_\varkappa}, D_{11}, \ldots, D_{\varkappa\rho_\varkappa})\, z_{\beta i}(t, \alpha_1, \ldots, \alpha_j)\, dt = 0$$

where $\Theta_{0\beta}^*$ are the components of the vector $\Theta(\phi^{(0)}, \xi^{(0)}, \xi^{(0)\cdot}, \xi^{(0)\cdot\cdot}, t, 0)$; and the feedback vector $\xi^{(0)}$ appearing here should be obtained as a function of the following constants, $A_{11}, \ldots, D_{\varkappa\rho_\varkappa}$ in accordance with the relations $\xi_s^{(0)} = (u^{(0)}, q_s)$ from (3.4).

The remaining $2h$ equations are obtained from the condition that the forms of the cosine and sine frequency components $\nu_n (n = 1, \ldots, \varkappa)$ of the right-hand side of the second Eq. of (3.3) should be orthogonal to all eigenvectors $u_{n1}, \ldots, u_{n\rho_n}$ corresponding to the value ν_n^2. We shall write these equations (assuming that all $\nu_n (n = 1, \ldots, \varkappa)$ enter (1.5); the terms containing ν_n and not appearing in (1.5), will simply not appear in (3.4)) in the form

$$P_{j+n+\rho}(\alpha_1, \ldots, \alpha_j, A_{n1}, \ldots, A_{n\rho_n}, D_{n1}, \ldots, D_{n\rho_n}) \equiv$$
$$\equiv \sum_{\beta=1}^{\rho_n} [((cC_1 - \nu_n^2 dM_1)\, u_{n\beta},\ u_{n\rho})\, A_{n\beta} + g\nu_n (Bu_{n\beta},\ u_{n\rho})\, D_{n\beta}] -$$
$$- f_1 \sum_{r=1}^{m} Q_{r\nu_n}^{(0)}(\alpha_1, \ldots, \alpha_j) \cos\vartheta_{r\nu_n}(\alpha_1, \ldots, \alpha_j)\,(q_r,\ u_{n\rho}) = 0$$

$$P_{j+n+\rho}^*(\alpha_1, \ldots, \alpha_j, A_{n1}, \ldots, A_{n\rho_n}, D_{n1}, \ldots, D_{n\rho_n}) \equiv$$
$$\equiv \sum_{\beta=1}^{\rho_n} [-g\nu_n (Bu_{n\beta},\ u_{n\rho})\, A_{n\beta} + ((cC_1 - \nu_n^2 dM_1)\, u_{n\beta},\ u_{n\rho})\, D_{n\beta}] -$$
$$- f_1 \sum_{r=1}^{m} Q_{r\nu_n}^{(0)}(\alpha_1, \ldots, \alpha_j) \sin\vartheta_{r\nu_n}(\alpha_1, \ldots, \alpha_j)\,(q_r,\ u_{n\rho}) = 0$$
$$(\rho = 1, \ldots, \rho_n;\ n = 1, \ldots, \varkappa) \tag{3.7}$$

The resonant procedure consists of constructing Eqs. (3.6) and (3.7), obtaining from them the parameters of the generating solution, etc.

We shall now consider an oscillating system characterized by a certain number of parameters (masses, rigidities, coefficients of friction etc.). We shall describe as resonant that part of the parametric space, in which (3.1) hold, and as nonresonant – that, in which the nonresonant assumptions given in Section 1 are valid. We shall write the matrices M and C as

$$M = M_0 + \delta M_1, \qquad C = C_0 + \sigma C_1 \tag{3.8}$$

The relations (1.10) hold in the nonresonant region

$$\alpha_i^* = \alpha_i^{**}(\ldots, fK_\nu, \Psi_\nu, \ldots) \qquad (i = 1, \ldots, j) \tag{3.9}$$

We note that the matrices K_ν and Ψ_ν exist also at some points of the resonant region when μ is sufficiently small (the case of $d = c = g = 0$ is excluded) and in this case we have $k_{\nu_n}^{(rs)} = O(1/\mu)$ and $fk_{\nu_n}^{(rs)} = O(1)$. Let us select a point in the resonant region and assume that functions α_i* are defined on some part of the space of their arguments and are continuous on it over the whole set. Let us also assume that the magnitudes $fk_{\nu_n}^{(rs)}$ ($n = 1,\ldots, \varkappa$; $r, s = 1,\ldots, m$) computed for the given point (in which $f = \mu f_1$, $\sigma = \mu c$, $\delta = \mu d$, $\gamma = \mu g$) belong to the region of definition of α_i^*. We shall determine α_i^* at the point in question of the resonant region using the relations (3.9).

Next we shall show that the magnitudes α_i^* thus defined, are connected with the magnitudes α_i^{**} obtained for the given point of the resonant region from the relations (3.6) and (3.7), by

$$\alpha_i^{**} = \alpha_i^* + O(\mu) \qquad (i = 1, \ldots, j) \tag{3.10}$$

With this purpose in mind, we shall consider the equation of forced oscillations of an oscillating system where the oscillations are excited by a load computed according to the generating solution for some $\alpha_1,\ldots, \alpha_j$

$$Mu^{(0)\cdot\cdot} + \gamma Bu^\cdot + Cu = f\sum_{r=1}^{m} Q_r(\varphi^{(0)}(t, \alpha))\, q_r \tag{3.11}$$

Eq. (3.11) admits a 2π-periodic solution at any point of the nonresonant region. We shall seek it in the form

$$u^{(0)} = \sum_{r=1}^{m} \sum_{\nu} (u_{\nu 1}^{(r)} \cos \nu t + u_{\nu_2}^{(r)} \sin \nu t) \tag{3.12}$$

Fourier coefficients $u_{\nu_1}^{(r)}$ and $u_{\nu_2}^{(r)}$ are obtained, in accordance with (1.5), from

$$(C - \nu^2 M)\, u_{\nu 1}^{(r)} \pm \gamma\nu B u_{\nu 2}^{(r)} = fQ_{r\nu}^{(0)} \cos \vartheta_{r\nu}\, q_r$$

$$-\gamma\nu B u_{\nu 1}^{(r)} + (C - \nu^2 M)\, u_{\nu 2}^{(r)} = fQ_{r\nu}^{(0)} \sin \vartheta_{r\nu}\, q_r \tag{3.13}$$

We shall seek the solution of (3.13) in the form of series in terms of the vectors $u_{11},\ldots, u_{\varkappa\rho_\varkappa}, u_{h+1}, \ldots, u_N$ (3.14)

$$u_{\nu 1}^{(r)} = \sum_{n=1}^{\varkappa} \sum_{\rho=1}^{\rho_n} v_{r\nu}^{(n,\rho)} u_{n\rho} + \sum_{l=h+1}^{N} v_{r\nu}^{(l)} u_l, \qquad u_{\nu 2}^{(r)} = \sum_{n=1}^{\varkappa} \sum_{\rho=1}^{\rho_n} w_{r\nu}^{(n,\rho)} u_{n\rho} + \sum_{l=h+1}^{N} w_{r\nu}^{(l)} u_l$$

which is possible, since these vectors form a basis in the configurational space of the oscillating system.

The following system of $2N$ linear algebraic equations yields the coefficients of (3.14):

$$\sum_{n=1}^{\varkappa} \sum_{\rho=1}^{\rho_n} [((C - \nu^2 M)\, u_{n\rho}, u_\eta)\, v_{r\nu}^{(n,\rho)} + \nu\gamma\, (Bu_{n\rho}, u_\eta)\, w_{r\nu}^{(n,\rho)}] + \tag{3.15}$$

$$+ \sum_{l=h+1}^{N} [((C - \nu^2 M)\, ul, u_\eta)\, v_{r\nu}^{(l)} + \nu\gamma\, (Bu_l, u_\eta)\, w_{r\nu}^{(l)}] - fQ_{r\nu}^{(0)} \cos \vartheta_{r\nu}\, (q_r, u_\eta) = 0$$

$$\sum_{n=1}^{\varkappa} \sum_{\rho=1}^{\rho_n} [-\nu\gamma\, (Bu_{n\rho}, u_\eta)\, v_{r\nu}^{(n,\rho)} + ((C - \nu^2 M)\, u_{n\rho}, u_\eta)\, w_{r\nu}^{(n,\rho)}] +$$

$$+ \sum_{l=h+1}^{N} [-\nu\gamma\, (Bu_l, u_\eta)\, v_{r\nu}^{(l)} + ((C - \nu^2 M)\, u_l, u_\eta)\, w_{r\nu}^{(l)}] - fQ_{r\nu}^{(0)} \sin \vartheta_{r\nu}\, (q_r, u_\eta) = 0$$

Here u_η is an eigenvector and Eqs. (3.15) are constructed for each of the N vectors $u_{11},\ldots, u_{\varkappa\rho_\varkappa}, \ldots, u_{h+1}, \ldots, u_N$.

All the systems of Eqs. (3.15) can be solved in the nonresonant region. If, for some particular point of this region, we use them to obtain $v_{r\nu}^{(n,\rho)}$ and $w_{r\nu}^{(n,\rho)}$ as functions of $\alpha_1,\ldots,\alpha_j$, obtain a solution of (3.11) from (3.14) and (3.12), compute for this solution the feedback parameters, insert them into small order terms of (1.1), take a mean value and find $\alpha_1^*,\ldots,\alpha_j^*$, then we shall find that the latter values can be obtained from (3.9), when fK_ν and Ψ_ν have values corresponding to the given point. The above systems have solutions at the previously chosen point of the resonant region, and the described sequence of operations yields the values of $\alpha_1^*,\ldots,\alpha_j^*$ corresponding to this point.

To see what form is assumed by (3.15) in the resonant region, we put $M = M_0 + \mu dM_1$, $C = C_0 + \mu cC_1$, $\gamma = \mu g$ and $f = \mu f_1$ and assume that ν in (3.15) is equal to some ν_n belonging to $\nu_1,\ldots,\nu_\varkappa$. Taking the orthonormalizing conditions (3.5) into account we obtain:

$$\sum_{\beta=1}^{\varkappa}\sum_{\rho=1}^{\rho_\beta}[((cC_1-\nu_n^2 dM_1)u_{\beta\rho},u_{ns})v_{r\nu_n}^{(\beta,\rho)}+\nu_n g(Bu_{\beta\rho},u_{ns})w_{r\nu_n}^{(\beta,\rho)}]+$$
$$+\sum_{l=h+1}^{N}[((cC_1-\nu_n^2 dM_1)u_l,u_{ns})v_{r\nu_n}^{(l)}+\nu_n g(Bu_l,u_{ns})w_{r\nu_n}^{(l)}]-$$
$$-f_1Q_{r\nu_n}^{(0)}\cos\vartheta_{r\nu_n}(q_r,u_{ns})=0\qquad (s=1,\ldots,\rho_n)\tag{3.16}$$

for the values of η in (3.15) corresponding to the vectors $u_{n1},\ldots,u_{n\rho_n}$ and

$$(\nu_\eta^2-\nu_n^2)v_{r\nu_n}^{(\eta)}+\mu\Big\{\sum_{\beta=1}^{\varkappa}\sum_{\rho=1}^{\rho_\beta}[((cC_1-\nu_n^2 dM_1)u_{\beta\rho},u_\eta)v_{r\nu_n}^{(\beta,\rho)}+$$
$$+\nu g(Bu_{\beta\rho},u_\eta)w_{r\nu_n}^{(\beta,\rho)}]+\sum_{l=h+1}^{N}[((cC_1-\nu_n^2 dM_1)u_l,u_\eta)v_{r\nu_n}^{(l)}+$$
$$+\nu g(Bu_l,u_\eta)w_{r\nu_n}^{(l)}]\Big\}-\mu f_1Q_{r\nu_n}^{(0)}\cos\vartheta_{r\nu_n}(q_r,u_\eta)=0\tag{3.17}$$

for the remaining values of η.

Only the "cosine" equations corresponding to the first Eq. of (3.15) are given in (3.16) and (3.17); "sine" equations have analogous form. All equations corresponding to the values of $\nu \overline{\in} (\nu_1,\ldots,\nu_\varkappa)$ will have the form of (3.17).

Relations (3.16) and (3.17) yield

$$v_{r\nu_n}^{(n,\rho)},w_{r\nu_n}^{(n,\rho)}=O(1)\qquad(\rho=1,\ldots,\rho_n,\ n=1,\ldots,\varkappa),\qquad v_{r\nu}^{(\eta)},\ \ w_{r\nu}^{(\eta)}=O(\mu)$$

for all remaining values of ν and η. Therefore, from (3.12) and (3.14), after performing the summation over r, we obtain

$$u^{(0)}=\sum_{n=1}^{\varkappa}\sum_{\rho=1}^{\rho_n}(v_{\nu_n}^{(n,\rho)}\cos\nu_n t+w_{\nu_n}^{(n,\rho)}\sin\nu_n t)u_{n\rho}+O(\mu)\tag{3.18}$$

Eqs. (3.16) and the corresponding sine equations, after the summation over r (see (3.7)), yield

$$\begin{gathered}P_{j+n+\rho}(\alpha_1,\ldots,\alpha_j,v_{\nu_n}^{(n,1)},\ldots,v_{\nu_n}^{(n,\rho_n)},w_{\nu_n}^{(n,1)},\ldots,w_{\nu_n}^{(n,\rho_n)})+O(\mu)=0\\ P_{j+n+\rho}^*(\ldots)+O(\mu)=0\qquad(\rho=1,\ldots,\rho_n,\ n=1,\ldots,\varkappa)\end{gathered}\tag{3.19}$$

and we finally obtain the following relations:

$$v_{\nu_n}^{(n,\rho)}(\alpha_1,\ldots,\alpha_j)=A_{n\rho}(\alpha_1,\ldots,\alpha_j)+O(\mu),\ w_{\nu_n}^{(n,\rho)}(\ldots)=D_{n\rho}(\ldots)\tag{3.20}$$

This proves the following. If we use the nonresonant procedure for a point belonging to the resonant region to obtain the feedback parameters as functions of $\alpha_1,\ldots,\alpha_j$ with the accuracy of the order of the small parameter, the result will be identical to that obtained, when the feedback parameters are computed according to the second Eq. of (3.4) in which $A_{n\rho}(\alpha_1,\ldots,\alpha_j)$ and $D_{n\rho}(\alpha_1,\ldots,\alpha_j)$ are obtained from (3.7). In other words, if the feedback $k_\nu^{(re)}$ and phase $\psi_\nu^{(re)}$ coefficients are computed for a point belonging to the resonant region and the relations (1.10) are used disregarding the fact that $k_{\nu_n}^{(re)} = O(1/\mu)$, then the resulting magnitudes $\alpha_1^*,\ldots,\alpha_j^*$ will coincide (with the accuracy of the order of μ) with the magnitudes $\alpha_1^{**},\ldots,\alpha_j^{**}$ obtained by means of the resonant procedure. This shows the correctness of (3.10) and it was established that (1.10) may be also applied to the resonant case.

Converse procedure – use of the resonant solution to determine nonresonant oscillations, is equivalent to retaining in the expansion of the solution of (3.12) into a Fourier series and in the expansions of the coefficients in terms of the eigenvectors u_η only those terms, which bring a contribution of the order of unity into the initial resonant region, and neglecting the remaining terms. If (1.5) does not contain the values of ν not appearing in (3.4) and the vectors q_r are linear combinations of the vectors $u_{n\rho}$ (the latter is obviously necessary for an oscillating system with one degree of freedom), then the resonant generating solution will coincide with the nonresonant one everywhere.

From this we can infer, that, if we only wish to construct a solution, then, for a system of the type of (1.1) the resonant case need not be considered separately. Comparison of the conditions of stability in both cases becomes interesting in this context. Let us take, for example, the problem of oscillations generated by a rotating unbalanced body. Conditions of stability obtained in [4] upon considering the vibrator as an almost conservative object and under nonresonant assumptions, coincide with the corresponding condition obtained by Kononenko in [3] by, what is in fact, a resonant procedure.

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Translated by L.K.